# FAMILIES OF ABELIAN VARIETIES WITH MANY ISOGENOUS FIBRES

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Abstract. Let $Z$ be a subvariety of the moduli space of principally polarised abelian varieties of dimension $g$ over the complex numbers. Suppose that a Zariski dense set of points of $Z$ correspond to abelian varieties from a single isogeny class. A generalisation of a conjecture of André and Pink predicts that $Z$ is a weakly special subvariety. We prove this when $\dim Z = 1$ using the Pila–Zannier method and the Masser–Wüstholz isogeny theorem. This generalises results of Edixhoven and Yafaev when the Hecke orbit consists of CM points and of Pink when it consists of Galois generic points.

## 1. Introduction

Let $\mathcal{A}_g$ denote the Siegel moduli space of principally polarised abelian varieties of dimension $g$. We consider the following conjecture. In particular we prove the conjecture when $Z$ is a curve, and make some progress on higher–dimensional cases.

**Conjecture 1.1.** *Let $\Lambda$ be the isogeny class of a point $s \in \mathcal{A}_g(\mathbb{C})$. Let $Z$ be an irreducible closed subvariety of $\mathcal{A}_g$ such that $Z \cap \Lambda$ is Zariski dense in $Z$. Then $Z$ is a weakly special subvariety of $\mathcal{A}_g$.*

**Theorem 1.2.** *Conjecture 1.1 holds when $Z$ is a curve.*

**Theorem 1.3.** *Let $\Lambda$ be the isogeny class of a point $s \in \mathcal{A}_g(\mathbb{C})$. Let $Z$ be an irreducible closed subvariety of $\mathcal{A}_g$ such that $Z \cap \Lambda$ is Zariski dense in $Z$.*

*Then there is a special subvariety $S \subset \mathcal{A}_g$ which is isomorphic to a product of Shimura varieties $S_1 \times S_2$ with $\dim S_1 > 0$, and such that*

$$Z = S_1 \times Z' \subset S$$

*for some irreducible closed subvariety $Z' \subset S_2$.*

Theorem 1.3, but not Theorem 1.2, depends on results concerning the hyperbolic Ax–Lindemann conjecture from preprints of Pila and Tsimerman [PT12] and of Ullmo [Ull12].

Conjecture 1.1 is a consequence of the Zilber–Pink conjecture on subvarieties of Shimura varieties [Pin05b], by a minor modification of [Pin05b] Theorem 3.3. It is slightly more general than the $S = \mathcal{A}_g$ case of the following conjecture of

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André and Pink, because the isogeny class of $s \in \mathcal{A}_g(\mathbb{C})$ is sometimes bigger than the Hecke orbit: by **isogeny class** we mean the set of points $t \in \mathcal{A}_g(\mathbb{C})$ such that the corresponding abelian variety $A_t$ is isogenous to $A_s$, with no condition of compatibility between isogeny and polarisations. On the other hand the **Hecke orbit** consists of those points for which there is a **polarised isogeny** between the principally polarised abelian varieties – that is, an isogeny $\phi \colon A_s \to A_t$ satisfying $\phi^* \lambda_t \in \mathbb{Z}\lambda_s$, where $\lambda_s$ and $\lambda_t$ are the polarisations. In the case of $\mathcal{A}_g$ there is no difference between Hecke orbits and Pink's generalised Hecke orbits.

**Conjecture 1.4.** *[*[And89] *Chapter X Problem 3,* [Pin05a] *Conjecture 1.6] Let $S$ be a mixed Shimura variety over $\mathbb{C}$ and $\Lambda \subset S$ the generalised Hecke orbit of a point $s \in S$. Let $Z \subset S$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in $Z$. Then $Z$ is a weakly special subvariety of $S$.*

Some cases of Conjecture 1.1 are already known: If the point $s$ is Galois generic, then the isogeny class and the Hecke orbit coincide, and Conjecture 1.1 follows from equidistribution results of Clozel, Oh and Ullmo, as was shown by Pink [Pin05a]. When $s$ is a special point, Theorem 1.2 was proved by Edixhoven and Yafaev [EY03] by counting intersections to show that $Z$ is contained in $T_g Z$ for some $g \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$. When $s$ corresponds to a product of elliptic curves, Habegger and Pila proved the theorem [HP12] using the method we extend here.

The notion of weakly special or totally geodesic subvarieties was introduced by Moonen [Moo98]. An algebraic subvariety $Z$ of $\mathcal{A}_g$ is called **weakly special** if there exists a sub-Shimura datum $(H, X_H)$ of $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$, a decomposition

$$(H^{\mathrm{ad}}, X_H) = (H_1, X_1) \times (H_2, X_2)$$

and a point $x_2 \in X_2$ such that $Z$ is the image in $\mathcal{A}_g$ of $X_1 \times \{x_2\}$. In other words, to say that $Z$ is weakly special means that we can choose $S$, $S_1$, $S_2$ in the conclusion of Theorem 1.3 such that $Z'$ is a single point in $S_2$.

In this article we will use a characterisation of weakly special subvarieties due to Ullmo and Yafaev [UY11]: $Z$ is weakly special if and only if an irreducible component of $\pi^{-1}(Z)$ is algebraic, where $\pi$ is the quotient map $\mathcal{H}_g \to \mathcal{A}_g$. Theorem 1.3 requires a strengthening of this characterisation called the hyperbolic Ax–Lindemann conjecture: if $W$ is a maximal algebraic subvariety of $\pi^{-1}(Z)$, then $\pi(W)$ is algebraic. A proof of the Ax–Lindemann conjecture for $\mathcal{A}_g$ was recently announced by Pila and Tsimerman [PT12].

Our proof of Theorems 1.2 and 1.3 follows the method proposed by Pila and Zannier for proving the Manin–Mumford and André–Oort conjectures [PZ08]. This is based upon counting rational points of bounded height in certain analytic subsets of $\mathcal{H}_g$, and applying the Pila–Wilkie counting theorem on sets definable in o-minimal structures.

The central part of the proof of Theorems 1.2 and 1.3 is in section 2. This uses a strong version of the Pila–Wilkie counting theorem involving definable blocks.

The other ingredients are an upper bound for the heights of matrices in $\mathrm{GL}_{2g}(\mathbb{Q})$ relating isogenous points, proved in section 3, and a lower bound for the Galois degrees of principally polarised abelian varieties in an isogeny class, derived from the Masser–Wüstholz isogeny theorem [MW93].

In section 4 we use a specialisation argument to prove a version of the Masser–Wüstholz isogeny theorem for finitely generated fields of characteristic 0, generalising the original theorem which was valid only over number fields. This is necessary in order to prove Theorems 1.2 and 1.3 for points $s$ and subvarieties $Z$ defined over $\mathbb{C}$ and not only $\bar{\mathbb{Q}}$.

Now we consider some generalisations of Theorem 1.2. Theorem 1.2 immediately implies Conjecture 1.4 for curves $Z$ in Shimura varieties $S$ of Hodge type, if we restrict to usual Hecke orbits. This is because, by the definition of a Shimura variety of Hodge type, there is a finite morphism $f\colon S \to \mathcal{A}_g$ for some $g$ such that the image of each Hecke orbit in $S$ is contained in a Hecke orbit of $\mathcal{A}_g$, and $Z \subset S$ is weakly special if and only if $f(Z) \subset \mathcal{A}_g$ is weakly special. However this does not imply Conjecture 1.4 for generalised Hecke orbits in Shimura varieties of Hodge type, as a generalised Hecke orbit may map into infinitely many isogeny classes in $\mathcal{A}_g$.

Thanks to the use of the Masser–Wüstholz theorem, our method applies only to Shimura varieties parameterising abelian varieties i.e. those of Hodge type. In particular, let us compare with Theorem 1.2 of [EY03]. Take any Shimura datum $(G, X)$. Edixhoven and Yafaev generalise Hecke orbits by choosing a representation of $G$ and considering a set of points where the induced $\mathbb{Q}$-Hodge structures are isomorphic. The Masser–Wüstholz theorem can be used only when these Hodge structures have type $(-1, 0) + (0, -1)$. Hence our method lacks a key advantage of Edixhoven and Yafaev's formulation, namely that they can replace $G$ by its adjoint.

This restriction to isogeny classes of abelian varieties rather than generalised Hecke orbits is related to our inability to prove the full Conjecture 1.1. In the case of the André–Oort conjecture, a conclusion as in Theorem 1.3 implies the full conjecture by induction on $\dim Z$ (see [Ull12]). This is because, when $Z$ is of the form $S_1 \times Z'$, special points in $Z$ project to special points in $Z'$.

This does not work for Conjecture 1.1 because the fact that $Z = S_1 \times Z'$ contains a dense set of isogenous points does not imply the same thing for $\{x_1\} \times Z'$, where we fix a point $x_1 \in S_1$ in order to realise $Z'$ as a subvariety of $\mathcal{A}_g$. The problem is that in order to see the decomposition $S = S_1 \times S_2$ at the level of Shimura data, we may have to use the adjoint Shimura datum and hence there is no interpretation of the decomposition in terms of the moduli of abelian varieties. For example this happens in Moonen's example of a subvariety $S \subset \mathcal{A}_8$ which decomposes as a product of Shimura varieties but where the generic abelian variety in the family parameterised by $S$ is simple.

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## 2. Proof of main theorem

In this section we will deduce our main theorems 1.2 and 1.3 from the matrix height bounds of section 3 and the isogeny bound of section 4. Accordingly fix a point $s \in \mathcal{A}_g(\mathbb{C})$ and let $\Lambda$ be its isogeny class. Let $Z \subset \mathcal{A}_g$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in $Z$.

We begin with some definitions and notation. Let $\pi \colon \mathcal{H}_g \to \mathcal{A}_g$ denote the quotient map and $\mathcal{F}_g$ a fundamental domain in $\mathcal{H}_g$ for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Let

$$\widetilde{Z} = \pi^{-1}(Z) \cap \mathcal{F}_g \quad \text{and} \quad \widetilde{\Lambda} = \pi^{-1}(\Lambda) \cap \mathcal{F}_g.$$

Fix a point $\tilde{s} \in \mathcal{H}_g$ such that $\pi(\tilde{s}) = s$.

We define the **complexity** of a point $t \in \Lambda$ to be the minimum degree of an isogeny $A_s \to A_t$ between the abelian varieties corresponding to the points $s$ and $t$ of $\mathcal{A}_g$. We may also talk about the complexity of a point in $\widetilde{\Lambda}$, meaning the complexity of its image in $\Lambda$.

We recall the definition of definable block from [Pil11]. Throughout this section, **definable** means definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. A **(definable) block** of dimension $w$ in $\mathbb{R}^\nu$ is a connected definable subset $W \subseteq \mathbb{R}^\nu$ of dimension $w$, regular at every point, such that there is a semialgebraic set $A \subseteq \mathbb{R}^\nu$ of dimension $w$, regular at every point, with $W \subseteq A$.

For a matrix $\gamma \in \mathrm{M}_{n \times n}(\mathbb{Q})$, the **height** $H(\gamma)$ will mean the maximum of the standard multiplicative heights of the entries of $\gamma$.

The key step in the proof of the main theorems is Proposition 2.1: the points of $\widetilde{Z} \cap \widetilde{\Lambda}$ of a given complexity are contained in subpolynomially many definable blocks, these blocks themselves contained in $\widetilde{Z}$. This is proved using the Pila–Wilkie counting theorem and the matrix height bounds of section 3.

On the other hand, the Masser–Wüstholz isogeny theorem gives a polynomial lower bound for the Galois degree of points in $\Lambda$ in terms of their complexity. Combining these two bounds, for large enough complexity there are more points in $\widetilde{Z} \cap \widetilde{\Lambda}$ than there are blocks to contain them. Hence most points of $\widetilde{Z} \cap \widetilde{\Lambda}$ are contained in blocks of positive dimension. In particular the union of positive-dimensional blocks contained in $\widetilde{Z}$ has Zariski dense image in $Z$.

In the case $\dim Z = 1$ this implies that $\widetilde{Z}$ has an algebraic irreducible component and so we can conclude using the Ullmo–Yafaev characterisation of weakly special subvarieties. When $\dim Z > 1$, we use the Ax–Lindemann theorem for $\mathcal{A}_g$ to deduce that positive-dimensional weakly special subvarieties are dense in $Z$ and then a result of Ullmo to complete the proof of Theorem 1.3.

**Proposition 2.1.** *Let $Z$ be a subvariety of $\mathcal{A}_g$ and $\tilde{s}$ a point in $\mathcal{H}_g$. Let $\epsilon > 0$.*

*There is a constant $c = c(Z, \tilde{s}, \epsilon)$ such that for every $n \geq 1$, there is a collection of at most $cn^\epsilon$ definable blocks $W_i \subset \widetilde{Z}$ such that the union $\bigcup W_i$ contains all points of $\widetilde{Z} \cap \widetilde{\Lambda}$ of complexity $n$.*

Let us outline the proof of this proposition. We cannot apply the counting theorem to $\widetilde{\Lambda} \subset \widetilde{Z}$ directly, because the points of $\widetilde{\Lambda}$ are transcendental. Instead we construct a definable set $Y$ with a semialgebraic map $\sigma\colon Y \to \widetilde{Z}$ such that points of $\widetilde{Z} \cap \widetilde{\Lambda}$ have rational preimages in $Y$, with heights polynomially bounded in terms of their complexity. This idea is due to Habegger and Pila [HP12].

Consider first the case $\operatorname{End} A_s = \mathbb{Z}$. This case is easier because all isogenies between $A_s$ and any abelian variety are polarised. In this case we let

$$Y = \{\gamma \in \mathrm{GSp}_{2g}(\mathbb{R})^+ \mid \gamma.\tilde{s} \in \widetilde{Z}\},$$

and let $\sigma\colon Y \to \widetilde{Z}$ be the map $\sigma(\gamma) = \gamma.\tilde{s}$.

Let $\tilde{t} \in \widetilde{Z} \cap \widetilde{\Lambda}$ and $t = \pi(\tilde{t})$. Then there is an isogeny $f\colon A_t \to A_s$ whose degree is equal to the complexity of $t$. By the hypothesis $\operatorname{End} A_s = \mathbb{Z}$ this isogeny is polarised. Hence the rational representation of $f$ (explained in section 3) gives a matrix $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ such that $\pi(\gamma.\tilde{s}) = t$ and whose height is polynomially bounded with respect to the complexity. We can also find $\gamma_1 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ of polynomially bounded height such that $\gamma_1\gamma.\tilde{s} = \tilde{t}$. Hence every point in $\widetilde{Z} \cap \widetilde{\Lambda}$ has a rational preimage in $Y$ of polynomially bounded height. This is precisely what we need to apply the Pila–Wilkie theorem to $Y$.

If we drop the assumption $\operatorname{End} A_s = \mathbb{Z}$ then this no longer works, because the rational representation of a non-polarised isogeny is not in $\mathrm{GSp}_{2g}(\mathbb{Q})^+$. Note that even if we assume that $t$ is in the Hecke orbit of $s$, so that there is some polarised isogeny $A_s \to A_t$, the isogeny of minimum degree need not be polarised. Thus we do not get an element of $\mathrm{GSp}_{2g}(\mathbb{Q})^+$ whose height is polynomially bounded in terms of the complexity.

To avoid this problem we will take $Y$ to be a subset of $\mathrm{GL}_{2g}(\mathbb{R})$ instead of $\mathrm{GSp}_{2g}(\mathbb{R})$. This will allow us to carry out the same proof using the rational representation of a not-necessarily-polarised isogeny. Of course $\mathrm{GL}_{2g}(\mathbb{R})$ does not act on $\mathcal{H}_g$ but this does not matter: the map

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\tilde{s} + b)(c\tilde{s} + d)^{-1} \quad \text{for } a, b, c, d \in \mathrm{M}_{g \times g}(\mathbb{R})$$

is defined on a Zariski open subset of $\mathrm{GL}_{2g}(\mathbb{R})$, and we will only consider matrices in $\mathrm{GL}_{2g}(\mathbb{R})$ where $\sigma$ is defined and has image in $\mathcal{H}_g$. In particular let

$$Y = \sigma^{-1}(\widetilde{Z}).$$

Before proving Proposition 2.1, we need to check that every element of $\widetilde{Z} \cap \widetilde{\Lambda}$ has a rational preimage in $Y$ whose height is polynomially bounded with respect

to the complexity. Proposition 3.1 says that there is some preimage in $\mathrm{GL}_{2g}(\mathbb{R})$ with this property, and all we need to do is move it into the fundamental domain.

**Lemma 2.2.** *There exist constants $c, k$ depending only on $g$ and $\tilde{s}$ such that:*

*For any $\tilde{t} \in \widetilde{Z} \cap \widetilde{\Lambda}$ of complexity $n$, there is a rational matrix $\gamma \in Y$ such that $\sigma(\gamma) = \tilde{t}$ and $H(\gamma) \leq cn^k$.*

*Proof.* Let $t = \pi(\tilde{t})$. Let $\mathcal{B}$ be a symplectic basis for $H_1(A_s, \mathbb{Z})$ with period matrix $\tilde{s}$.

By Proposition 3.1 there is an isogeny $f \colon A_t \to A_s$ and a symplectic basis $\mathcal{B}'$ for $H_1(A_t, \mathbb{Z})$ such that the rational representation $\gamma_1$ of $f$ has polynomially bounded height. As remarked in the introduction to section 3, $\sigma({}^t\gamma_1)$ is the period matrix for $(A_t, \lambda_t)$ with respect to the basis $\mathcal{B}'$. In particular,

$$\pi\sigma({}^t\gamma_1) = t.$$

Hence there is $\gamma_2 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\gamma_2.\sigma({}^t\gamma_1) = \tilde{t}$. By [PT11] Lemma 3.4, $H(\gamma_2)$ is polynomially bounded. Then $\gamma = \gamma_2{}^t\gamma_1$ satisfies the required conditions. □

Now we are ready to prove Proposition 2.1. We simply apply the Pila–Wilkie theorem to $Y$, using Lemma 2.2 to relate heights of rational points in $Y$ to complexities of points in $\widetilde{Z} \cap \widetilde{\Lambda}$. We then use the fact that $\sigma$ is semialgebraic, and that the blocks in $Y$ can be chosen uniformly from finitely many definable families, to go from $Y$ to $\widetilde{Z}$.

*Proof of Proposition 2.1.* The set

$$Y = \sigma^{-1}(\pi^{-1}(Z) \cap \mathcal{F}_g)$$

is definable because $\sigma$ is semialgebraic and $\pi_{|\mathcal{F}_g}$ is definable by a theorem of Peterzil and Starchenko [PS11].

Hence we can apply the Pila–Wilkie theorem, in the form of [Pil11] Theorem 3.6, to $Y$: for every $\epsilon > 0$, there are finitely many definable block families $\mathcal{W}^{(j)}(\epsilon) \subset Y \times \mathbb{R}^\lambda$ and a constant $c_1(Y, \epsilon)$ such that for every $T \geq 1$, the rational points of $Y$ of height at most $T$ are contained in the union of at most $c_1 T^\epsilon$ definable blocks $W_i(T, \epsilon)$, taken from the families $\mathcal{W}^{(j)}(\epsilon)$.

Since $\sigma$ is semialgebraic, the image under $\sigma$ of a definable block in $Y$ is a finite union of definable blocks in $\widetilde{Z}$. Furthermore the number of blocks in the image is uniformly bounded in each definable block family $\mathcal{W}^{(j)}(\epsilon)$. Hence $\sigma(\bigcup W_i(T, \epsilon))$ is the union of at most $c_2 T^\epsilon$ blocks in $\widetilde{Z}$, for some new constant $c_2(Z, \tilde{s}, \epsilon)$.

But by Lemma 2.2, for suitable constants $c, k$, every point of $\widetilde{Z} \cap \widetilde{\Lambda}$ of complexity $n$ is in $\sigma(\bigcup W_i(cn^k, \epsilon))$. □

Proposition 2.1 tells us that the points of $\widetilde{Z} \cap \widetilde{\Lambda}$ of complexity $n$ are contained in fewer than $c(\epsilon)n^\epsilon$ blocks for every $\epsilon > 0$. On the other hand, the Masser–Wüstholz isogeny theorem implies that the number of such points grows at least as fast as $n^{1/k}$ for some constant $k$. Hence most points of $\widetilde{Z} \cap \widetilde{\Lambda}$ are contained in a block of positive dimension. We check that this is sufficient, with the hypothesis that

$Z \cap \Lambda$ is Zariski dense in $Z$, to deduce that the union of positive-dimensional blocks in $\widetilde{Z}$ has Zariski dense image in $Z$.

**Proposition 2.3.** *Let $\Lambda_1$ be the set of points $t \in Z \cap \Lambda$ for which there is a positive-dimensional block $W \subset \widetilde{Z}$ such that $t \in \pi(W)$.*

*If $Z \cap \Lambda$ is Zariski dense in $Z$, then $\Lambda_1$ is Zariski dense in $Z$.*

*Proof.* Let $Z_1$ denote the Zariski closure of $\Lambda_1$ (a priori this could be empty).

Let $(A_s, \lambda_s)$ be a polarised abelian variety corresponding to the point $s \in \mathcal{A}_g(\mathbb{C})$, defined over a finitely generated field $K$. We choose $K$ large enough that the varieties $Z$ and $Z_1$ are also defined over $K$.

Let $t$ be a point in $Z \cap \Lambda$ of complexity $n$. The polarised abelian variety corresponding to $t$ might not have a model over the field of moduli $K(t)$, but it has a model $(A_t, \lambda_t)$ over an extension $L$ of $K(t)$ of uniformly bounded degree – the degree $[L : K(t)]$ is at most $|\mathrm{Sp}_{2g}(\mathbb{Z}/3)|$.

By Theorem 4.1, the complexity $n$ is bounded above by a polynomial $c[L : K]^k$ in $[L : K]$, with $c$ and $k$ depending only on $A_s$ and $K$. Hence for a different constant $c_1$, we have

$$[K(t) : K] \geq c_1 n^{1/k}.$$

But all $\mathrm{Gal}(\bar{K}/K)$-conjugates of $t$ are contained in $Z \cap \Lambda$ and have complexity $n$. By Proposition 2.1, the preimages in $\mathcal{F}_g$ of these points are contained in the union of $c_2(Z, \tilde{s}, 1/2k)n^{1/2k}$ definable blocks, each of these blocks being contained in $\widetilde{Z}$.

For large enough $n$, we have

$$c_1 n^{1/k} > c_2 n^{1/2k}.$$

For such $n$, by the pigeonhole principle there is a definable block $W \subset \widetilde{Z}$ such that $\pi(W)$ contains at least two Galois conjugates of $t$. Since blocks are connected by definition, $\dim W > 0$. So those conjugates of $t$ in $\pi(W)$ are in $\Lambda_1$. Since $Z_1$ is defined over $K$, it follows that $t$ itself is also in $Z_1$.

In other words all points of $Z \cap \Lambda$ of large enough complexity are in $Z_1$. But this excludes only finitely many points of $Z \cap \Lambda$. So as $Z \cap \Lambda$ is Zariski dense in $Z$, we conclude that $Z_1 = Z$. $\square$

Call a subset $W \subset \mathcal{H}_g$ **complex algebraic** if it is the connected component of $W_0 \cap \mathcal{H}_g$ for some positive-dimensional irreducible complex algebraic varieties $W_0 \subset \mathrm{M}_{2g \times 2g}(\mathbb{C})$. Let $\widetilde{Z}^{\mathrm{ca}}$ denote the complex algebraic part of $\widetilde{Z}$ – that is, the union of positive-dimensional complex algebraic subsets of $\mathcal{H}_g$ contained in $\widetilde{Z}$.

By Lemma 2.1 of [Pil09], $\widetilde{Z}^{\mathrm{ca}}$ is the same as the union of the definable blocks contained in $\widetilde{Z}$. So Proposition 2.3 tells us that $\pi(\widetilde{Z}^{\mathrm{ca}})$ is Zariski dense in $Z$.

If $\dim Z = 1$, then the fact that $\widetilde{Z}^{\mathrm{ca}}$ is non-empty implies that some irreducible component of $\widetilde{Z}$ is complex algebraic. By [UY11] this implies that $Z$ is weakly special, proving Theorem 1.2.

For $\dim Z > 1$, we use the Ax–Lindemann theorem for $\mathcal{A}_g$ proved by Pila and Tsimerman [PT12]: if $W$ is a maximal complex algebraic subset of $\widetilde{Z}$ then $\pi(W)$ is weakly special. Hence $\pi(\widetilde{Z}^{\mathrm{ca}})$ is a union of positive-dimensional weakly special subvarieties, so these are dense in $Z$. Let $S$ be the smallest special subvariety of $\mathcal{A}_g$ containing $Z$. By Théorème 1.3 of [Ull12], we deduce that $S = S_1 \times S_2$ for some Shimura varieties $S_1$ and $S_2$, and $Z = S_1 \times Z'$ for some subvariety $Z \subset S_2$, proving Theorem 1.3.

## 3. Heights of rational representations of isogenies

Let $(A, \lambda)$ and $(A', \lambda')$ be principally polarised abelian varieties over $\mathbb{C}$ related by an isogeny of degree $n$ (not necessarily compatible with the polarisations). In this section we show that, for suitable choices of bases for $H_1(A, \mathbb{Z})$ and $H_1(A', \mathbb{Z})$ and of isogeny $f \colon A' \to A$, the height of the rational representation of $f$ is polynomially bounded in $n$. In particular we prove the following proposition, used to prove Lemma 2.2.

**Proposition 3.1.** *Let $(A, \lambda)$ be a principally polarised abelian variety over $\mathbb{C}$ and fix a symplectic basis $\mathcal{B}$ for $H_1(A, \mathbb{Z})$. There exist constants $c, k$ depending only on $(A, \lambda)$ such that:*

*If $(A', \lambda')$ is any principally polarised abelian variety for which there exists an isogeny $A \to A'$ of degree $n$, then there are an isogeny $f \colon A' \to A$ and a symplectic basis $\mathcal{B}'$ for $H_1(A', \mathbb{Z})$ such that*

$$H(f, \mathcal{B}', \mathcal{B}) \leq cn^k.$$

We define the **rational representation** of an isogeny $f \colon A' \to A$ (with respect to bases $\mathcal{B}, \mathcal{B}'$ for $H_1(A, \mathbb{Z})$ and $H_1(A', \mathbb{Z})$) to be the matrix of the induced morphism

$$f_* \colon H_1(A', \mathbb{Z}) \to H_1(A, \mathbb{Z})$$

in terms of the chosen bases. This gives a $2g \times 2g$ integer matrix. We write

$$H(f, \mathcal{B}', \mathcal{B})$$

for the height of the rational representation of $f$, meaning simply the maximum of the absolute values of the entries of the matrix.

Rational representations of isogenies are particularly interesting in the case that the bases $\mathcal{B}, \mathcal{B}'$ are symplectic with respect to the polarisations $\lambda, \lambda'$. In this case, if $\tilde{s}, \tilde{t} \in \mathcal{H}_g$ are the period matrices of $(A, \lambda)$ and $(A', \lambda')$ with respect to the chosen bases and $\gamma$ is the rational representation of an isogeny $A' \to A$, then

$$\tilde{t} = (a\tilde{s} + b)(c\tilde{s} + d)^{-1} \text{ where } {}^t\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

We remark also that for symplectic bases, an isogeny is polarised if and only if its rational representation is in $\mathrm{GSp}_{2g}(\mathbb{Q})$.

In Proposition 3.1, the isogeny whose existence is assumed and the isogeny whose existence is asserted in the conclusion go in opposite directions. This is the

most convenient formulation for our application, but it is not important since any isogeny $A \to A'$ of degree $n$ gives rise to an isogeny in the opposite direction of degree $n^{2g-1}$.

In order to prove the proposition, let $h\colon A \to A'$ be an isogeny of degree $n$. Then $h^*\lambda'$ is a polarisation of $A$, so there is an endomorphism $q \in \operatorname{End} A$ such that

$$h^*\lambda' = \lambda \circ q.$$

Furthermore $q$ is **symmetric** i.e. $q^\dagger = q$ (where $\dagger$ is the Rosati involution with respect to $\lambda$) and **positive definite** i.e. each component of $q$ in

$$\operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod \mathrm{M}_{l_i}(\mathbb{R}) \times \prod \mathrm{M}_{m_i}(\mathbb{C}) \times \prod \mathrm{M}_{n_i}(\mathbb{H})$$

has positive eigenvalues.

We can identify $H_1(A', \mathbb{Z})$ with a submodule of $H_1(A, \mathbb{Z})$ of index $n^{2g-1}$, and so find a basis for $H_1(A', \mathbb{Z})$ whose height is at most $n^{2g-1}$. However this need not be a symplectic basis. We apply the standard algorithm for finding a symplectic basis: the height of this new basis is controlled by $h^*\lambda'$, in other words by $q$.

So we would like to bound the height of the rational representation of $q$ in terms of $\deg h$. However this is not possible: let $A$ be an abelian variety whose endomorphism ring is the ring of integers $\mathfrak{o}$ of a real quadratic field. In particular $\mathfrak{o}$ has infinitely many units. Let $h$ be a unit in $\mathfrak{o}$ – in other words, an isomorphism $A \to A$. If we take the same polarisation on each copy of $A$, then $q = h^2$ and the rational representation of this can have arbitrarily large height.

We can avoid this by replacing $h$ by $h \circ u$ for some automorphism $u$ of $A$ – recall that all we have supposed about $h$ is that it is an isogeny $A \to A'$ of degree $n$. This replaces $q$ by $u^\dagger q u$. We will show that we can choose $u$ so that the height of the rational representation of $u^\dagger q u$ is bounded by a multiple of $\deg q = n^2$.

The following proposition is motivated by the theorem [Mil86] that the symmetric elements of $\operatorname{End} A$ of a given norm fall into finitely many orbits under the action of $(\operatorname{End} A)^\times$ given by $(u, q) \mapsto uqu^\dagger$. In geometric terms, this says that if we fix $A$ and $\deg \mu$ then there are finitely many isomorphism classes of polarised abelian varieties $(A, \mu)$. Our proposition strengthens this by saying that each orbit contains an element whose height is bounded by a multiple of the norm. Milne's theorem is proved using the reduction theory of arithmetic groups. We also use reduction theory, but in order to get height bounds we have to go deeper into the structure of $\operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{R}$.

The representation $\rho$ appears in the proposition solely to give us a convenient definition of heights and norms of elements of $R$. Specifically, $H(x)$ means the height of $\rho(x)$ and $\mathrm{N}(x) = \det \rho(x)$ for $x \in R$.

**Proposition 3.2.** *Let $(E, \dagger)$ be a semisimple $\mathbb{Q}$-algebra with a positive involution, let $R$ be a $\dagger$-stable order in $E$ and let $\rho : R \to \mathrm{M}_N(\mathbb{Z})$ be a faithful representation of $R$.*

*There is a constant $c$ depending only on $(R, \dagger, \rho)$ such that for any symmetric positive definite $q \in R$, there is some $u \in R^\times$ such that*

$$H(u^\dagger q u) \leq c \, \mathrm{N}(q).$$

*Proof.* We begin by checking that it suffices to prove the proposition for simple algebras $E$. In general, $E = \prod E_i$ for some simple $\mathbb{Q}$-algebras $E_i$. Let $R_i = R \cap E_i$. Then $R' = \prod R_i$ is an order of $E$ contained in $R$. Let $m = [R : R']$. Given $q \in R$, we look at $mq \in R'$. Suppose that the proposition holds for each $R_i$; then clearly it holds for $R'$, so there is $u \in R'^\times$ (a fortiori $u \in R^\times$) such that

$$H(umqu^\dagger) \leq c \, \mathrm{N}(mu).$$

Hence the proposition holds for $R$ with constant $c \, \mathrm{N}(m)/m$.

So we suppose that $E$ is simple. Then $E = \mathrm{M}_n(D)$ for some division algebra $D$, and the involution $\dagger$ is matrix transposition composed with some involution of $D$. We may also suppose that $R$ is contained in the maximal order $\mathrm{M}_n(\mathfrak{o})$, where $\mathfrak{o}$ is a maximal order in $D$.

By the Albert classification, $E \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to one of $\mathrm{M}_{nd}(\mathbb{R})^r$, $\mathrm{M}_{nd}(\mathbb{C})^r$ or $\mathrm{M}_{nd}(\mathbb{H})^r$. Because $q$ is symmetric, its projection onto each simple factor of $E \otimes_{\mathbb{Q}} \mathbb{R}$ is a Hermitian matrix. By the theory of Hermitian forms over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, there exist $x, d \in E \otimes_{\mathbb{Q}} \mathbb{R}$ such that $d$ is diagonal with real entries in each factor and

$$q = x^\dagger d x.$$

Since $q$ is positive definite, all the diagonal entries of $d$ are positive so we can multiply each row of $x$ by the square root of the corresponding entry of $d$ to suppose that $d = 1$. We then have $q = x^\dagger x$.

Let $G$ be the $\mathbb{Z}$-group scheme

$$G(A) = (R \otimes_{\mathbb{Z}} A)^\times.$$

Over $\mathbb{Q}$ this is the reductive group $\mathrm{Res}_{D/\mathbb{Q}} \mathrm{GL}_n$. We will use the following notations for subgroups of $G$:

(i) $S$ is the maximal $\mathbb{Q}$-split torus of $G$ whose $\mathbb{Q}$-points are the diagonal matrices of $\mathrm{GL}_n(D)$ with entries in $\mathbb{Q}$;
(ii) $P$ is the minimal parabolic $\mathbb{Q}$-subgroup of $G$ consisting of upper triangular matrices;
(iii) $U = R_u(P)$ is the group of upper triangular matrices with ones on the diagonal;
(iv) $M$ is the maximal $\mathbb{Q}$-anisotropic subgroup of $Z(S)^0$; that is, $M(\mathbb{Q})$ consists of the diagonal matrices in $\mathrm{GL}_n(D)$ whose diagonal entries have reduced norm $\pm 1$;
(v) $K = \{g \in G(\mathbb{R}) \mid g^\dagger g = 1\}$ is a maximal compact subgroup of $G(\mathbb{R})$.

By Proposition 13.1 of [Bor69], there exist a positive real number $t$, a finite set $C \subset G(\mathbb{Q})$ and a compact neighbourhood $\omega$ of 1 in $M^0(\mathbb{R})U(\mathbb{R})$ such that

$$G(\mathbb{R}) = K A_t \omega C G(\mathbb{Z})$$

where

$$A_t = \{a \in S(\mathbb{R}) \mid a_i > 0, a_i/a_{i+1} \leq t \text{ for all } i\}.$$

We note that $M^0(\mathbb{R})U(\mathbb{R})$ is the group of upper triangular matrices in $\mathrm{M}_n(D \otimes_{\mathbb{Q}} \mathbb{R})$ whose diagonal entries have reduced norm 1.

Hence we can write

$$x = kaz\nu\gamma$$

where $k \in K$, $a \in A_t$, $z \in \omega$, $\nu \in C$ and $\gamma \in G(\mathbb{Z}) = R^\times$.

Let $u = \gamma^{-1}$ and

$$q' = u^\dagger q u.$$

In order to prove the proposition, it will suffice to show that $H(q') \leq c\,\mathrm{N}(q)$.

Since $k^\dagger k = 1$, and using the decomposition of $x$, we get that

$$q' = \nu^\dagger z^\dagger a^\dagger a z \nu.$$

Fix some $\mathbb{Z}$-basis of $R$. We will show below that the (real) coordinates of $a^\dagger a$ are bounded by a constant multiple of $\mathrm{N}(q)$. The coordinates of $z$ and $\nu$ are uniformly bounded because $z$ is in the compact set $\omega$ and $\nu$ is in the finite set $C$. Hence the coordinates of $q'$ in this basis are bounded by a multiple of $\mathrm{N}(q)$, so $H(q')$ is likewise linearly bounded.

Let $a^\dagger a = \mathrm{diag}(a_1, \dots, a_n)$ with $a_i \in \mathbb{R}$. In order to show that the coordinates of $a^\dagger a$ in the chosen basis are bounded, it will suffice to show that the $a_i$ are bounded by a multiple of $\mathrm{N}(q)$. The product $\prod a_i$ is at most a multiple of $\mathrm{N}(q)$, so to show that the $a_i$ are bounded above we will show that they are bounded below by a constant.

Choose an integer $m$ such that $m\nu^{-1} \in R$ for all $\nu \in C$. Then

$$m^2 z^\dagger a^\dagger a z = (m\nu^{\dagger -1}) q' (m\nu^{-1}) \in R$$

so every entry of $m^2 z^\dagger a^\dagger a z$, viewed as a matrix in $\mathrm{M}_n(D)$, is in $\mathfrak{o}$.

Let $z_{11}$ denote the upper left entry of $z \in \mathrm{M}_n(D \otimes_{\mathbb{Q}} \mathbb{R})$. Because $z$ is upper triangular, the upper left entry of $m^2 z^\dagger a^\dagger a z$ is $m^2 z_{11}^\dagger a_1 z_{11}$. So $m^2 z_{11}^\dagger a_1 z_{11} \in \mathfrak{o}$ and

$$\left|\mathrm{Nrd}_{D/\mathbb{Q}}(m^2 z_{11}^\dagger a_1 z_{11})\right| \geq 1.$$

But $\mathrm{Nrd}(z_{11}) = 1$ because $z \in \omega$, so

$$\left|\mathrm{Nrd}_{D \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{R}}(m^2 a_1)\right| \geq 1.$$

Since $m^2 a_1$ is a positive real number, $\mathrm{Nrd}_{D \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{R}}(m^2 a_1)$ is just some fixed positive power of $m^2 a_1$ so we conclude that

$$m^2 a_1 \geq 1.$$

From the definition of $A_t$, it follows that $a_i \geq m^{-2} t^{2-2i}$ for all $i$ and we have established that the $a_i$ are uniformly bounded below.

Hence there is a constant $c_1$ such that for every $j$,

$$a_j \leq c_1 \prod a_i.$$

Since $\rho$ is faithful $\dim \rho \geq n$. Together with the fact that $\prod a_i$ is bounded below this implies that

$$\prod a_i \leq c_2 \left(\prod a_i\right)^{\dim \rho / n} = c_2 \, \mathrm{N}(a^\dagger a).$$

Now $\mathrm{N}(z) = \mathrm{N}(u) = 1$ and $\mathrm{N}(\nu)$ is bounded because $\nu$ comes from a finite set, so $\mathrm{N}(a^\dagger a)$ is bounded above by a constant multiple of $\mathrm{N}(q)$. Combining all this we have proved that each $a_i$ is bounded above by a constant multiple of $\mathrm{N}(q)$, and as remarked above this suffices to establish the proposition. □

We will need the following bound for the height of a symplectic basis for a symplectic free $\mathbb{Z}$-module in terms of the values of the symplectic pairing on the standard basis. The proof is simply to apply the standard recursive algorithm for finding a symplectic basis, verifying that the new vectors introduced always have polynomially bounded heights.

**Lemma 3.3.** *Let $L = \mathbb{Z}^{2g}$ and let $\{e_1, \dots, e_{2g}\}$ be a basis for $L$. There exist constants $c, k$ depending only on $g$ such that:*

*For any perfect symplectic pairing $\psi : L \times L \to \mathbb{Z}$ with*

$$N = \max_{i,j} \left|\psi(e_i, e_j)\right|,$$

*there exists a symplectic basis for $(L, \psi)$ whose coordinates with respect to the basis $\{e_1, \dots, e_{2g}\}$ are at most $cN^k$.*

*Proof.* For any $x \in L$, we write $H(x)$ for the maximum of the absolute values of the coordinates of $x$ with respect to the basis $\{e_1, \dots, e_{2g}\}$.

First let $e_1' = e_1$ and choose $e_2'$ such that $\psi(e_1', e_2') = 1$ and $H(e_2') \leq N$. We can do this because $\psi$ is perfect, so that $\gcd_{i=2}^n(\psi(e_1, e_i)) = 1$. Hence there are integers $a_i$ such that $|a_i| \leq N$ and

$$\sum a_i \psi(e_1, e_i) = 1$$

We let $e_2' = \sum a_i e_i$.

Then find $e_3', \dots, e_{2g}'$ orthogonal to $e_1'$ and to $e_2'$ such that $\{e_1', \dots, e_{2g}'\}$ is a basis for $L$ and $H(e_i') \leq 2gN^2$. We can do this by setting

$$e_i' = e_i + \psi(e_2', e_i)e_1' + \psi(e_1', e_i)e_2'.$$

Here we have $|\psi(e_2', e_i)| \leq \sum_{j=2}^n |a_j \psi(e_j, e_i)| \leq (2g-1)N^2$ and $\psi(e_1', e_i)e_2'$ has height at most $N^2$ so $H(e_i') \leq 2gN^2$.

Finally apply the algorithm recursively to $L' = \mathbb{Z}\langle e_3', \dots, e_{2g}' \rangle$. We have

$$\left|\psi(e_i', e_j')\right| \leq gNH(e_i')H(e_j') \leq 4g^3N^5.$$

Hence by induction $L'$ has a symplectic basis whose coordinates with respect to $\{e_3', \dots, e_{2g}'\}$ are bounded by a constant multiple of $N^{5k'}$, where $k'$ is the exponent in the lemma for $\mathbb{Z}^{2(g-1)}$. Converting these into coordinates with respect to

$\{e_1, \ldots, e_{2g}\}$, we get that the elements of this symplectic basis for $L'$ have height bounded by a constant multiple of $N^{2+5k'}$. This proves the lemma.

We remark that the recurrence $k(g) = 2 + 5k(g-1)$, $k(0) = 0$ is satisfied by $k(g) = (5^g - 1)/2$. $\square$

*Proof of Proposition 3.1.* Let $h\colon A \to A'$ be an isogeny of degree $n$. There is $q \in \operatorname{End} A$ such that

$$h^*\lambda' = \lambda \circ q.$$

Apply Proposition 3.2 to get $u \in (\operatorname{End} A)^\times$ such that

$$H(u^\dagger q u) \leq c\,\mathrm{N}(q).$$

Then $hu$ is an isogeny $A \to A'$ of degree $n$, so there is also an isogeny $f\colon A' \to A$ of degree $n^{2g-1}$ such that

$$hu \circ f = [n]_A.$$

The image of $f_*\colon H_1(A', \mathbb{Z}) \to H_1(A, \mathbb{Z})$ is a submodule of index $n^{2g-1}$. By the structure theory of finitely generated $\mathbb{Z}$-modules there is a basis $\{e'_1, \ldots, e'_{2g}\}$ for $H_1(A', \mathbb{Z})$ with respect to which the rational representation of $f$ is upper triangular and has height at most $n^{2g-1}$. But this need not be a symplectic basis.

Let $\psi, \psi'$ be the symplectic forms on $H_1(A, \mathbb{Z})$ and $H_1(A', \mathbb{Z})$ induced by $\lambda, \lambda'$ respectively. Let $q' = u^\dagger q u$. Then

$$n^2\lambda' = [n]^*_{A'}\lambda' = f^*u^*h^*\lambda' = f^*(\lambda \circ q').$$

In terms of symplectic forms this says that

$$n^2\psi'(x, y) = \psi(f_*x, q'_* f_* y).$$

In particular, since the coordinates (with respect to $\mathcal{B}$, a symplectic basis for $\psi$) of $\{f_*e'_1, \ldots, f_*e'_{2g}\}$ and the entries of the matrix $q'_*$ are bounded by a polynomial in $n$, the same is true for

$$\left|\psi'(e'_i, e'_j)\right|.$$

Hence by Lemma 3.3 there is a symplectic basis $\mathcal{B}'$ for $H_1(A', \mathbb{Z})$ whose coordinates with respect to $\{e'_1, \ldots, e'_{2g}\}$ are polynomially bounded. Using again that the coordinates with respect to $\mathcal{B}$ of $\{f_*e'_1, \ldots, f_*e'_{2g}\}$ are polynomially bounded, we deduce that $H(f, \mathcal{B}', \mathcal{B})$ is also polynomially bounded. $\square$

## 4. Isogeny theorem over finitely generated fields

The Masser–Wüstholz isogeny theorem [MW93] gives a bound for the minimum degree of an isogeny between two abelian varieties over number fields, as a function of one of the varieties and the degree of their joint field of definition. In order to prove Theorems 1.2 and 1.3 for points $s \in \mathcal{A}_g$ defined over $\mathbb{C}$ and not merely over $\bar{\mathbb{Q}}$, we need to extend the isogeny theorem to abelian varieties defined over finitely generated fields of characteristic 0. We will do this by a specialisation argument,

using the fact that any abelian scheme has a closed fibre in which the specialisation map of endomorphism rings is surjective.

A key feature of the theorem of Masser and Wüstholz is the explicit dependence of the bound on the abelian variety $A$, via the Faltings height. Our theorem does not make this explicit, and it is not apparent that there is any analogy of the Faltings height over a finitely generated field which would enable it to be made explicit. Instead what matters to us is the dependence on the field of definition of $B$.

**Theorem 4.1.** *Fix an integer $\delta$ and be a finitely generated field $K$ of characteristic 0. Let $A$ be an abelian variety defined over $K$ with a polarisation of degree at most $\delta$. There exist constants $c(A, K, \delta)$ and $\kappa$ ($\kappa$ depending only on $\dim A$) such that:*

*If $B$ is any abelian variety defined over a finite extension $L$ of $K$, with a polarisation of degree at most $\delta$ and isogenous to $A$, then there exists an isogeny $A \to B$ of degree at most*

$$c(A, K, \delta)[L : K]^\kappa.$$

In this theorem, polarisations and isogenies are defined over $\bar{K}$ and we do not require isogenies to be compatible with the polarisation.

*Proof.* Let $R$ be a finitely generated normal $\mathbb{Q}$-algebra whose field of fractions is $K$, and let $S = \operatorname{Spec} R$. There is an abelian scheme $\mathcal{A}$ over some open subset $U \subset S$ whose generic fibre is isomorphic to $A$.

By replacing $L$ by a larger extension of bounded degree (the bound depending only on $\dim A$), we may assume that all homomorphisms $A \to B$ are defined over $L$. Let $R'$ be the integral closure of $R$ in $L$ and $S' = \operatorname{Spec} R'$. Let $\pi : S' \to S$ be the obvious finite morphism and let $U' = \pi^{-1}(U)$.

Because $A$ and $B$ are isogenous, there is an abelian scheme $\mathcal{B}$ over $U'$ with generic fibre isomorphic to $B$, and such that $\mathcal{B}$ is isogenous to $\mathcal{A}$. We can construct this as follows: let $N$ be the kernel of an isogeny $A \to B$. We can extend $N$ to a finite flat subgroup scheme $\mathcal{N} \subset \mathcal{A}$. Then let $\mathcal{B}$ be the quotient $\mathcal{A}/\mathcal{N}$.

For any closed points $s' \in U'$ and $s = \pi(s') \in U$, the fibres $\mathcal{A}_s$ and $\mathcal{B}_{s'}$ are abelian varieties over the number fields $k_s$ and $k_{s'}$, isogenous over $k_{s'}$. We can apply the Masser–Wüstholz theorem to deduce that there are constants $c(\mathcal{A}_s, k_s, \delta)$ and $\kappa(\dim A)$ and an isogeny $\mathcal{A}_s \to \mathcal{B}_{s'}$ of degree at most

$$c(\mathcal{A}_s, k_s, \delta)[k_{s'} : k_s]^\kappa.$$

Observe that $[k_{s'} : k_s] \leq [L : K]$.

In order to prove the theorem, all we have to do is show that this isogeny $\mathcal{A}_s \to \mathcal{B}_{s'}$ lifts to an isogeny $A \to B$ (which will have the same degree). Hence it will suffice to show that there is some closed point $s$ such that the specialisation map

$$\operatorname{Hom}_{\bar{K}}(A, B) \to \operatorname{Hom}_{\bar{k}_s}(\mathcal{A}_s, \mathcal{B}_{s'}) \tag{*}$$

is surjective. Because we want a bound which depends only on $A$ and not on $B$, we have to show that there is a single point $s \in U$ which will work for all $B$.

We choose a closed point $s \in U$ such that $\operatorname{End}_{\bar{K}} A \to \operatorname{End}_{\bar{k_s}} \mathcal{A}_s$ is surjective. Such an $s$ exists by [Noo95] Corollary 1.5 (this is proved using the Hilbert irreducibility theorem).

Let $f_s$ be a $\bar{k}_s$-homomorphism $\mathcal{A}_s \to \mathcal{B}_{s'}$. To prove that (*) is surjective, we have to show that $f_s$ lifts to $\bar{K}$-homomorphism $A \to B$.

We are assuming that $A$ and $B$ are isogenous. Choose any isogeny $g_\eta : A \to B$ and let $g_s$ be its specialisation at $s$. Let

$$\alpha_s = g_s^{-1} \circ f_s \in \operatorname{End}_{\bar{k_s}} \mathcal{A}_s \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By our choice of $s$, this lifts to some $\alpha_\eta \in \operatorname{End}_{\bar{K}} A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $f_\eta = g_\eta \circ \alpha_\eta$ is a quasi-isogeny $A \to B$ specialising to $f_s$.

All we have to do is check that $f_\eta$ is an isogeny and not just a quasi-isogeny. Choose an integer $m$ such that $mf_\eta$ is an isogeny. The kernel of $mf_s$ contains $\mathcal{A}_s[m]$ so lifting to the generic fibre, the kernel of $mf_\eta$ contains $A[m]$. Hence $mf_\eta$ factorises as $f'_\eta \circ [m]$ for an isogeny $f'_\eta : A \to B$, and we must have $f'_\eta = f_\eta$. □

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